JOURNAL OF COMPUTATIONAL PHYSICS 3, 291–306 (1968)

Asymptotic Fitting, a Method for Solving Anisotropic Transfer Problems in Thick Layers

H. C. VAN DE HULST

Sterrewacht, Leiden, Netherlands

Received June 21, 1968

ABSTRACT

Practical solutions of the reflection and transmission of radiation from very thick and semi-infinite homogeneous atmospheres with an arbitrary phase function are found as follows. Numerical data for three values of the thickness, computed by the doubling method, are fitted to rigorous asymptotic expressions. An automatic fitting procedure yields the constants and functions occurring in these expressions. These include the escape function, i.e., the solution of the corresponding Milne problem, the diffusion length and the extrapolation length. Results for isotropic scattering and for the Henyey–Greenstein phase functions are presented. The accuracy is excellent for single-scattering albedos $\geqslant 0.6$.

1. PROBLEM AND METHOD

Light from a distant source, e.g. the Sun, falling on a cloud consisting of well separated water drops can be scattered many times in succession by different drops before ever emerging from the cloud. The problem to find the diffuse radiation field thus set up is the prototype of a multiple scattering or radiative transfer problem. If the cloud has the form of a homogeneous slab and the source illuminates it in a certain direction from above, the problem is to find the radiation emerging from the top of the layer (reflection) and from the bottom (transmission). Since the drops are far apart, only their far-field scattering pattern enters into the computation; random displacements wash out all interference effects. The basic given function therefore represents the combined intensity scattering pattern of all drops in a volume element; the traditional name for this function is the phase function.

A thin slab is said to have the optical thickness $d\tau$ if a parallel beam shining perpendicularly through this slab is transmitted with $1 - d\tau$ of its original intensity. It is said to have an albedo a if a fraction a of the lost radiation reappears in the form of scattered radiation. The remaining fraction $1 - a$ is absorbed in the scattering particles, converted into heat and its energy may reappear in a quite different, infrared, wavelength region, which we shall not consider. We assume that the cloud is statistically homogeneous. Further we assume that the incident and scattered light is unpolarized, although this is not essential to the method described below.

The properties of the cloud now are fully characterized by the phase function $\Phi(\cos \alpha)$ and albedo a of a volume element and by its total optical thickness b. We seek to determine the reflection function $R(\mu, \mu_0, \varphi - \varphi_0)$ and transmission function $T(\mu, \mu_0, \varphi - \varphi_0)$, each of which is symmetric in μ and μ_0 and even in $\varphi - \varphi_0$. Here μ_0 is the cosine of the angle of incidence, μ the cosine of the angle of emergence, both reckoned positive, and φ and φ_0 are azimuth angles. For simplicity we restrict our attention in this paper to the averages of these functions over $\varphi - \varphi_0$, i.e., to the azimuth-independent terms $R(\mu, \mu_0)$ and $T(\mu, \mu_0)$. The problem posed in this paper hardly arises for the other terms, anyhow, since the terms dependent on azimuth are damped out more rapidly in thick layers.

Mathematically identical problems are encountered in neutron scattering. The restriction to a fixed wavelength band in optical scattering corresponds to the one-velocity assumption in neutron scattering. Numerous problems to find the entire radiation field, or only the reflection and transmission functions, have been devised and numerically worked out. Three books [1, 2, 3] may be cited for a full review.

Most of the published numerical results refer to rather simple phase functions. Among these are isotropic scattering, linearly anisotropic scattering and scattering with a phase function expandible in Legendre functions up to $N = 2$. More strongly anisotropic phase functions, such as occur in natural haze and clouds, have made the numerical treatment of the transfer problems notoriously difficult, particularly for very thick atmospheres. The practical choice often was between crude treatment of the transfer problems with the desired phase function or exact treatment of the transfer problem with a very crude phase function.

This paper poses no limitation on the form of the phase function. The method and results described here are based on the availability of

(a) Expressions for the reflection and transmission by a homogeneous, plane-parallel atmosphere with large optical thickness b and arbitrary scattering pattern, which are asymptotically exact in the limit $b \to \infty$. Most of these were first given by Germogenova [4]. They will be used here in the form derived from physical principles by Van de Hulst [5]. These formulae may be considered as asymptotic forms (for large optical depth) in which only the dominant terms of the full solution by Case's method are retained. For a mathematical discussion of related problems compare ([3], [6], [7], [8]).

(b) The doubling method, which permits a simple and accurate computation of these functions for any finite b. This method, suggested by Van de Hulst and Irvine [9], was later used and described in some detail by a number of authors ([10], [11], [12]). In a different context the same method had already been given by Peebles and Plesset [13]).

The "asymptotic fitting method" described in this paper is a sequel to (a) and (b). It consists of pushing the doubling method (b) far enough, say, to numerical results for $b = 8$, 16 and 32, and then choose the unknown functions and constants occurring in the expressions (a) to fit these numbers. This may be the fastest method yet available to calculate the reflection functions (in the terminology of neutron scattering, the "albedo problem") and the emerging radiation (i.e., the "Milne problem") for a semi-infinite atmosphere with an arbitrary nearly-conservative phase function.

2. Scheme of Computation

We employ the notation of [5] but write all integrals fully, instead of in the shorthand matrix notation employed there. The given function is the phase function $\Phi(\cos\alpha)$ with albedo a and anisotropy factor g defined by

$$a = \frac{1}{2}\int_{-1}^{1} \Phi(x)\,dx$$

$$ag = \frac{1}{2}\int_{-1}^{1} \Phi(x)\,x\,dx$$

The functions to be determined are $R_\infty(\mu, \mu_0)$ = reflection function, $K(\mu)$ = escape function, $P(u)$ = diffusion pattern, and the constants k, m, l, q occurring in Eqs. 3–6 below. Here μ_0 (cosine of angle of incidence) and μ (cosine of angle of emergence) range from 0 to 1, whereas u (the cosine of the angle between the direction of propagation and the downward vertical) has the full range from -1 to $+1$.

A homogeneous plane-parallel atmosphere of sufficient optical thickness b then has the following properties:

If $a < 1$ (nonconservative case)

Reflection function:

$$R(b, \mu, \mu_0) = R_\infty(\mu, \mu_0) - \frac{mf}{1 - f^2}\,e^{-kb}K(\mu)\,K(\mu_0) \tag{3a}$$

Transmission function:

$$T(b,\ \mu,\ \mu_0) = \frac{m}{1-f^2}\, e^{-kb} K(\mu)\, K(\mu_0) \tag{4a}$$

Intensity at midlayer, $\tau = \frac{1}{2}b$:

$$I(b,\ \tfrac{1}{2}b,\ u,\ \mu_0) = \frac{1}{1-f^2}\,[P(u) - fP(-u)]\, e^{-kb/2} K(\mu_0) \tag{5a}$$

Here

$$f = le^{-kb} = e^{-k(b+2q)} \tag{6}$$

Incidence is supposed to occur on the horizontal top surface with a flux π per unit area.

If $a = 1$ *(conservative case)* the corresponding formulae are:

$$R(b,\ \mu,\ \mu_0) = R_\infty(\mu,\ \mu_0) - T(b,\ \mu,\ \mu_0) \tag{3b}$$

$$T(b,\ \mu,\ \mu_0) = \frac{4K(\mu)\, K(\mu_0)}{3(1-g)(b+2q_0)} \tag{4b}$$

$$I(b,\ \tfrac{1}{2}b,\ u,\ \mu_0) = \left[\tfrac{1}{2} + \frac{u}{(1-g)(b+2q_0)}\right] K(\mu_0) \tag{5b}$$

The equations (3a)–(5a) and (3b)–(5b) are asymptotically correct for $b \to \infty$. Suppose we have obtained by the doubling method full numerical data for $b = b_0$, $2b_0$, $4b_0$ and that b_0 is chosen large enough for equations (3)–(5) to hold with good numerical accuracy. We may then by subsequent solution of two quadratic equations solve for the unknown constants and functions.

The fitting process thus achieved is written down below in the form of a recipe which may be taken as the basis of a computer program. The recipe for the non-conservative case can be verified from equations (3a)–(5a) if we know that $z = \exp(-kb_0)$, $\gamma = z + z^{-1}$, and $\delta = lz^2$. An enormous amount of redundancy can be built into the program because many expressions are derived as functions of b_0, or μ, or μ_0, or all three which should be asymptotically independent of these variables. If b_0 is not chosen large enough, small variations with the redundant parameter are noticeable. If b_0 is chosen too large, cumulative errors or round-off errors may affect the numerical accuracy.

The speed of the method cannot easily be assessed in an objective manner. The asymptotic fitting process adds an insignificant amount of computing time to the time necessary for executing the doubling method, say to $b = 32$. This may, therefore, seem a bit wasteful, *if* only the asymptotic results are sought. However, the conceptual simplicity, and the many internal checks may often make this method more attractive than a computation based on the more abstract concepts of Case's method.

Recipe for the nonconservative case

Choose a large b_0 and find consecutively

step		Remark:
101	$\alpha = T(b_0, \mu, \mu_0)/T(2b_0, \mu, \mu_0)$	1
102	$\beta = T(2b_0, \mu, \mu_0)/T(4b_0, \mu, \mu_0)$	1
103	$\gamma = (2\alpha)^{-1} + [(4\alpha^2)^{-1} + \beta + 1]^{1/2}$	7
104	$z = \frac{1}{2}\gamma - (\frac{1}{4}\gamma^2 - 1)^{1/2}$	7
105	$\delta = \dfrac{R(4b_0, \mu, \mu_0) - R(2b_0, \mu, \mu_0)}{T(2b_0, \mu, \mu_0) - z^2 T(4b_0, \mu, \mu_0)}$	1
106	$l = \delta z^{-2}$	
107	$k = -(\ln z)/b_0$	3
108	$q = -(\ln l)/2k$	
109	$R_\infty(\mu, \mu_0) = R(2b_0, \mu, \mu_0) + \delta T(2b_0, \mu, \mu_0)$	5
110	$mK(\mu)\, K(\mu_0) = (1 - \delta^2)\, z^{-2} T(2b_0, \mu, \mu_0)$	5
111	$M_0(2b_0, \mu_0) = \int_{-1}^{1} I(2b_0, b_0, u, \mu_0)\, du \qquad [= \mathrm{ND} + \mathrm{NA}]$	4, 5
112	$M_1(2b_0, \mu_0) = \int_{-1}^{1} I(2b_0, b_0, u, \mu_0)\, u\, du \qquad [= \frac{1}{2}\mathrm{UD} - \frac{1}{2}\mathrm{NA}]$	4, 5
113	$K(\mu_0) = (1 + \delta)\, M_0(2b_0, \mu_0)/2z$	5
114	$k^{-1}K(\mu_0) = (1 - \delta)\, M_1(2b_0, \mu_0)/2(1 - a)\, z$	5
115	$k = K(\mu_0)/[k^{-1}K(\mu_0)]$	2, 3
116	$m = [mK(\mu)\, K(\mu_0)]/K(\mu)\, K(\mu_0)$	1
117	$P(u) = [I(2b_0, b_0, u, \mu_0) + \delta I(2b_0, b_0, -u, \mu_0)]/zK(\mu_0)$	2, 5

Recipe for the conservative case

Choose a large b and find

		Remark:
201	$R_\infty(\mu, \mu_0) = R(b, \mu, \mu_0) + T(b, \mu, \mu_0)$	5
202	$t_{11}(b) = \int_0^1 \int_0^1 T(b, \mu, \mu_0)\, \mu\mu_0\, d\mu\, d\mu_0 \qquad [= \frac{1}{4}\mathrm{UTU}]$	4

203	$2q_0 = [3(1-g)\,t_{11}]^{-1} - b$		
204	$t_1(b, \mu_0) = \int_0^1 T(b, \mu, \mu_0)\,\mu\,d\mu$	$[=\tfrac{1}{2}\mathrm{UT}]$	4, 5
205	$K(\mu) = T(b, \mu, \mu_0)/2t_1(b, \mu_0)$		2, 5
206	$2q_0 = b\{[T(b, \mu, \mu_0)/T(2b, \mu, \mu_0) - 1]^{-1} - 1\}$		1, 6

Remarks:

1. These ratios of two functions of μ and μ_0 should be independent of μ and μ_0; this provides many checks. The same ratio should be obtained if numerator and denominator are replaced by integrals over μ or μ_0, or both. In practice we used integrals with the weights $4\mu\mu_0\,d\mu\,d\mu_0$ to find these ratios; in the matrix notation [5] these were written as URU and UTU.

2. Same as 1, but here numerator and denominator are functions of one variable μ_0, which drops out in taking the ratio.

3. These two formulae for k provide a sensitive check; formula 107 is the more accurate one.

4. In these equations the corresponding expression in matrix notation is given in square brackets.

5. Any of these functions of μ and μ_0, or of μ_0, or of μ, can be integrated to find its moments. We can find these moments also directly by starting to use the corresponding moment at the righthand side. In our doubling program, the moments and bimoments of R, T and I were always given with the function.

6. This independent way to derive $2q_0$ is added here for those who wish to avoid using the bi-moments t_{11}.

7. Both these equations are solutions of quadratic equations. Except in the case of very nearly conservative scattering, where z is close to 1, a quicker solution of these equations is obtained by expansion. We employ the fact that $z \ll 1$ and hence $\delta \ll 1$.

Appropriate formulae are:

$$
\begin{aligned}
z^{-1} &= \beta^{1/2} + \left(\frac{1}{2} + \frac{3}{4\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta^{1/2}}\right) + O(z^7) \\
z &= \beta^{-1/2} - \tfrac{1}{2}\beta^{-1}(\alpha^{-1} - \beta^{-1/2}) + O(z^7)
\end{aligned}
\tag{6}
$$

3. Results for Isotropic Scattering

The reflection and escape functions for isotropic scattering ($g = 0$) depend by well-known formulae on a single H-function:

$$R_\infty(\mu, \mu_0) = aH(\mu)\, H(\mu_0)/4(\mu + \mu_0) \tag{7}$$

$$K(\mu) = K(0)H(\mu)/(1 - k\mu) \tag{8}$$

As customary, the dependence of H, K and k on a is not explicitly written. These equations provide additional internal checks as well as checks against H-functions tabulated elsewhere. In the conservative case ($a = 1$) the same equations hold with $k = 0$, $K(0) = \frac{1}{4}\sqrt{3}$.

We have executed the asymptotic fitting program described in the preceding section for $g = 0$ as we did for any other g. From the functions thus found, $H(\mu)$ was computed in three different ways, based on Eqs. (7) or (8):

$$H(\mu) = [8\mu R_\infty(\mu, \mu)/a]^{1/2} \tag{9}$$

$$H(\mu) = 4\mu R_\infty(\mu, 0)/a \tag{10}$$

$$H(\mu) = (1 - k\mu)\, K(\mu)/K(0) \tag{11}$$

The computation was made for $\mu = 0.1, 0.3, 0.5, 0.7, 0.9, 1$ and $a = 0.6, 0.8, 0.9, 0.95, 0.99, 1$. A table is not presented, because the numbers from (9) and (10) always checked within 1×10^{-6} with the 6-decimal tables of Stibbs and Weir [14]. The values from (11) had the same accuracy for $a = 1$ and $a = 0.99$ but started to deviate in the fifth decimal at $a = 0.9$ and $a = 0.95$ and more strongly for lower a-values.

Isotropic scattering offers further checks on the quantities k, m and l. These functions of a can be solved (for $g = 0$) from the equations

$$\frac{1}{a} = \frac{1}{2k} \ln \frac{1 + k}{1 - k} \tag{12}$$

$$\frac{4}{m} = \frac{k(1 - k^2)}{a(a - 1 + k^2)} = -\frac{dk}{da} \tag{13}$$

$$f(0) = \frac{a}{2} \int_0^1 \frac{H(\mu)\, d\mu}{1 + k\mu} \tag{14}$$

$$K(0) = (a/m)\, [1 - f(0)] \tag{15}$$

$$l = (2m/ak)[K(0)]^2 = (2a/km)[1 - f(0)]^2 \tag{16}$$

Equations (12) and (13) have been known since many years. Eqs. (14)–(16) form a transcription of earlier results [15] into the notations employed here. Known relations of the H-functions make it possible to express also the moments of $K(\mu)$ in closed forms:

$$K_0 = \int_0^1 K(\mu)\, d\mu = (2/a)\, K(0) \qquad [= \mathrm{NK}] \tag{17}$$

$$K_1 = \int_0^1 K(\mu)\, \mu\, d\mu = (2/ak)(1-a)^{1/2}\, K(0) \qquad [= \tfrac{1}{2}\mathrm{UK}] \tag{18}$$

Again, the matrix notation for these integrals is given in square brackets. Finally, we can transform (16) into

$$l = akmK_1^2/2(1-a) \tag{19}$$

Table 1 shows some comparisons made by means of these formulae. The asymptotic fitting results were obtained by means of the scheme in Section 2, taking $b_0 = 16$. The agreement for $a = 0.8$ and higher is excellent. This was further confirmed by comparing the extrapolation length, q, found by asymptotic fitting to the exact values tabulated by Heaslet and Warming [16]. For $a = 0.6$ the differences are of the order of one percent.

The asymptotic fitting method is not accurate for $a < 0.6$, where k gets too close to 1. This is to be expected, for the derivation is based on the assumption that the factor $\exp(-b)$ can be neglected but $\exp(-kb)$ cannot. Table I suggests that fair accuracy is reached with the asymptotic fitting method if $k \leqslant 0.9$ and excellent results if $k \leqslant 0.7$.

TABLE I

CHECK CALCULATIONS FOR ISOTROPIC SCATTERING

Quantity	Method and equation	$a = 0.4$	0.6	0.8	0.9	0.95	0.99	1
k	fitting, step 115	.99768	.90865	.71043	.52543	.37949	.17251	0
	fitting, step 107	.98774	.90736	.71041	.52543	.37949	.17251	0
	exact, Eq. (12)	.98562	.90733	.71041	.52543	.37949	.17251	0
m	fitting, step 116	14.8520	6.21443	2.77038	1.66646	1.09974	.46751	0
	exact, Eq. (13)	21.1240	6.33416	2.77083	1.66646	1.09974	.46751	0
l	fitting, step 106	.02621	.11628	.28279	.43617	.56687	.78068	1
	fitting, Eq. (19)	.02470	.11513	.28275	.43617	.56687	.78068	1
	exact, Eqs. (14), (16)	.02740	.11492	.28275	.43617	.56688	.78069	1
$K(0)$	fitting, step 113	.02085	.07102	.17030	.24877	.30482	.37762	.43301
	exact, Eqs. (14), (15)	.01599	.07027	.17029	.24877	.30482	.37762	.43301

The value of $K(0)$ for isotropic, conservative scattering is 0.43301. For the same case we find by asymptotic fitting the extrapolation length $q_0 = 0.7104435$, as compared with the exact value 0.710446089800 (K. Grossman, private communication). The difference of 2.6×10^{-6} may be due to the limited number of gauss points in the integration.

4. Results for Henyey–Greenstein Phase Functions

Like other authors [17], [18], [12] we adopted the Henyey–Greenstein phase functions

$$\Phi(g, a, \cos\alpha) = \frac{a(1-g^2)}{(1+g^2-2g\cos\alpha)^{3/2}} \tag{20}$$

as a convenient set to experiment with numerical methods. These functions are completely determined by the parameters a = albedo and g = anisotropy factor; they permit a smooth transition from $g = 0$ (isotropic scattering) to $g = 1$ (complete forward scatter) or $g = -1$ (complete backscatter).

Presenting full data is impossible here, for not only are new tables required for each combination (g, a) but also the great simplification arising in isotropic scattering by the relation (7) does not have a simple equivalent for arbitrary phase functions. Hence $R_\infty(a, g, \mu, \mu_0)$ is truly a function of four variables with no simplifying property, but its symmetry in μ and μ_0.

A selection of numerical results is presented in tables 2–4. Some comments on these tables will now be given. Unfortunately, there is little in the literature which may serve for a direct check on the accuracy of these data, except for the checks pertaining to $g = 0$, which have already been discussed.

Table II presents the "constants", i.e. the functions which do not depend on direction (μ_0 or μ). The values of k and m were checked against numbers obtained by solving an integral equation. We found k as the smallest eigenvalue of the equation

$$(1 - ku)\, P(u) = \frac{1}{2}\int_{-1}^{1} h(u, v)\, P(v)\, dv \tag{21}$$

where

$$h(u, v) = \frac{1}{2\pi}\int_0^{2\pi} \Phi\{uv + (1-u^2)^{1/2}(1-v^2)^{1/2}\cos\varphi\}\, d\varphi \tag{22}$$

Normalizing the corresponding eigenfunction $P(u)$ by

$$\frac{1}{2}\int_{-1}^{1} P(u)\, du = 1 \tag{23}$$

we then found m by [5]

$$m = \int_{-1}^{1} [P(u)]^2 \, 2u \, du \tag{24}$$

TABLE II

VARIATION OF THE NUMBERS k, m, AND q, AND SOME COMBINATIONS, WITH THE ALBEDO a AND ANISOTROPY g OF THE HENYEY–GREENSTEIN PHASE FUNCTION. ALL NUMBERS WERE OBTAINED BY ASYMPTOTIC FITTING

Tabulated function	a	isotropic $g = 0$	0.25	0.5	waterdrops 0.75	0.875
k	1	0	0	0	0	0
	.99	.1725	.1496	.1224	.0871	.0623
	.95	.3795	.3310	.2733	.1990	.1480
	.9	.5254	.4618	.3856	.2881	.2224
	.8	.7104	.6344	.5418	.4234	.3448
m	1	0	0	0	0	0
	.99	.468	.539	.660	.935	1.327
	.95	1.100	1.263	1.545	2.197	3.148
	.9	1.666	1.900	2.317	3.303	4.678
	.8	2.771	3.087	3.719	5.291	7.671
$(1 - a)(1 - g)\,k^{-2}$	1	.3333	.3333	.3333	.3333	.3333
	.99	.3361	.3350	.3336	.3295	.3220
	.95	.3470	.3423	.3346	.3157	.2854
	.9	.3622	.3517	.3363	.3012	.2528
	.8	.3963	.3726	.3406	.2789	.2103
$m(1 - g)\,k^{-1}$	1	2.667	2.667	2.667	2.667	2.667
	.99	2.710	2.704	2.697	2.684	2.663
	.95	2.898	2.862	2.826	2.760	2.660
	.9	3.172	3.087	3.004	2.866	2.680
	.8	3.900	3.649	3.432	3.124	2.781
$(1 - g)\,q$	1	.7104	.7109	.7120	.7134	.7140
	.99	.7176	.7167	.7158	.7123	.7040
	.95	.7479	.7409	.7317	.7089	.6696
	.9	.7896	.7735	.7528	.7066	.6369
	.8	.8890	.8481	.8000	.7075	.5900

The values of k found from Eq. (21) showed no differences (in the decimals presented here) with the values found from asymptotic fitting shown in Table II.

The values of m found from Eq. (24) deviated by the following percentages from those shown in Table II:

	$g = 0.75$	$g = 0.875$
$a = 0.9$	0.01 %	0.2 %
$a = 0.8$	0.03 %	0.6 %
$a = 0.6$	0.3 %	2.1 %

TABLE III

SAMPLE VALUES OF THE ESCAPE FUNCTION $K(a, g, \mu)$ FOR THE HENYEY–GREENSTEIN PHASE FUNCTION. TWO MOMENTS ARE ALSO GIVEN. ALL NUMBERS WERE OBTAINED BY ASYMPTOTIC FITTING

Tabulated function		isotropic $g = 0$	0.25	0.5	large waterdrops 0.75	0.875
$K(a, g, 1)$ (perpendicular escape)	$a = 1$	1.2591	1.2613	1.2653	1.2697	1.2713
	0.99	1.1284	1.1101	1.0828	1.0258	.9556
	0.95	1.0204	.9830	.9353	.8550	.7790
	0.9	.9698	.9190	.8619	.7799	.7134
	0.8	.9399	.8638	.7942	.7151	.6626
$K(a, g, 0)$ (grazing escape)	$a = 1$	.4330	.4242	.3951	.3326	.2722
	0.99	.3776	.3635	.3280	.2566	.1889
	0.95	.3048	.2885	.2507	.1784	.1142
	0.9	.2488	.2335	.1975	.1305	.0748
	0.8	.1703	.1599	.1306	.0774	.0377
moment K_0 [= NK]	$a = 1$	.8660	.8644	.8599	.8524	.8427
	0.99	.7629	.7486	.7231	.6709	.6090
	0.95	.6417	.6207	.5824	.5060	.4226
	0.9	.5528	.5320	.4913	.4105	.3271
	0.8	.4257	.4129	.3769	.3023	.2291
moment $2K_1$ [= UK]	$a = 1$	1	1	1	1	1
	0.99	.8844	.8695	.8446	.7922	.7270
	0.95	.7563	.7322	.6921	.6127	.5254
	0.9	.6654	.6404	.5966	.5121	.4246
	0.8	.5360	.5181	.4781	.3984	.3191
$K(1, g, \mu)$ (conservative scattering)	$\mu = 0$	.4330	.4242	.3951	.3326	.2722
	0.1	.5401	.5341	.5160	.4842	.4628
	0.3	.7112	.7082	.7009	.6921	.6880
	0.5	.8716	.8705	.8691	.8686	.8689
	0.7	1.0280	1.0285	1.0303	1.0331	1.0348
	0.9	1.1824	1.1841	1.1876	1.1917	1.1936
	1	1.2591	1.2613	1.2653	1.2697	1.2713

The differences are $\leqslant 0.01\%$ in all other entries. We conclude that a lack of accuracy of the asymptotic fitting method just makes itself felt in the lower right corner of the table. Similar inaccuracy may be present in Tables III and IV but we have no direct check.

Two further combinations in Table II are presented. They approach for $a \to 1$ and arbitrary g the limits $(1 - a)(1 - g)k^{-2} \to 1/3$ and $m(1 - g)k^{-1} \to 8/3$. It is

TABLE IV

REFLECTION FUNCTION $R(a, g, 1, \mu)$ OF A SEMI-INFINITE ATMOSPHERE WITH HENYEY–GREENSTEIN PHASE-FUNCTION, CHARACTERIZED BY THE PARAMETERS a AND g, AND PERPENDICULAR INCIDENCE. THE FIRST MOMENT IS ALSO GIVEN. ALL NUMBERS HAVE BEEN OBTAINED BY ASYMPTOTIC FITTING.

Tabulated function	a	isotropic $g = 0$	0.25	0.5	large waterdrops 0.75	0.875
$R(a, g, 1, 0)$	1	.7270	.6829	.5774	.4389	.3439
	.99	.6120	.5559	.4377	.2847	.1824
	.95	.4933	.4328	.3155	.1743	.0943
	.9	.4163	.3578	.2489	.1255	.0625
	.8	.3196	.2693	.1782	.0825	.0392
$R(a, g, 1, 0.1)$	1	.8243	.7889	.7107	.6190	.5721
	.99	.6815	.6294	.5283	.3941	.2967
	.95	.5360	.4760	.3691	.2325	.1441
	.9	.4436	.3842	.2827	.1610	.0910
	.8	.3309	.2783	.1927	.0986	.0517
$R(a, g, 1, 0.5)$	1	.9755	.9682	.9581	.9486	.9437
	.99	.7542	.7171	.6598	.5557	.4416
	.95	.5498	.4951	.4164	.2897	.1807
	.9	.4318	.3730	.2937	.1795	.0975
	.8	.3012	.2451	.1762	.0921	.0439
$R(a, g, 1, 1)$	1	1.0569	1.0717	1.0950	1.1188	1.1309
	.99	.7567	.7279	.6840	.5786	.4480
	.95	.5123	.4582	.3863	.2595	.1489
	.9	.3851	.3244	.2516	.1437	.0692
	.8	.2554	.1963	.1352	.0633	.0263
$\int_0^1 R(a, g, 1, \mu)\, 2\mu\, d\mu$	1	1	1	1	1	1
	.99	.7527	.7183	.6646	.5592	.4397
	.95	.5355	.4809	.4039	.2768	.1677
	.9	.4149	.3555	.2778	.1655	.0866
	.8	.2853	.2282	.1615	.0815	.0375

evident that within the range of a and g values shown, and barring the entries near the lower right corner, these numbers and hence k and m themselves can easily be found in 3 or 4 figures by interpolation. The final entry in Table II is the extrapolation length q, again for interpolation purposes multiplied by $(1 - g)$.

Some data about the escape function are presented in Table III. In order to save space we have presented only the extreme values at grazing angles ($\mu = 0$) and in the normal direction ($\mu = 1$), as well as two moments. Only for conservative scattering values of $K(a, g, \mu)$ in a number of intermediate directions are given.

Values of the reflection function $R(\mu, \mu_0)$ presented in Table IV are limited to perpendicular incidence, $\mu_0 = 1$. Other angles of incidence would not only require further tables of $R(\mu, \mu_0)$, which we did obtain by asymptotic fitting, but also tables of the azimuth-dependent terms, which we have not yet computed. Again, one moment is presented. This is UR in matrix notation and is called the albedo of the half-space in the terminology of neutron scattering. Its value is 1 for a conservative atmosphere ($a = 1$).

5. Discussion

Upon scanning Tables II–IV we are struck first of all by the very small changes in all lines referring to conservative scattering, except when grazing angles are involved. Obviously, the reflection on and escape from a semi-infinite atmosphere are very similar for all values of g. There is a strong suggestion that finite limits for $g \to 1$ exist but this has not been established as a fact. Van de Hulst and Grossman [12] have shown that a close similarity exists for finite conservative slabs of optical thickness b if $b(1 - g)$ is kept constant and for finite non-conservative slabs if, in addition, $\gamma = (1 - a)/k$ is kept constant. The present tables provide further illustrations of these similarity laws.

The fact that all numerical examples are based on one special set of phase functions may have enhanced the similarity. Phase functions not belonging to this set might behave rather differently. In order to check this suspicion, the results for the Henyey–Greenstein function obtained here may be compared to those for phase functions with only a few terms in the expansion in terms of Legendre functions:

$$\Phi(\cos \alpha) = \sum_{n=0}^{N} \omega_n P_n(\cos \theta) \tag{25}$$

Many transfer problems with such truncated phase functions with $N = 0$ (isotropic scattering), 1, or 2 have been solved exactly ([1], [19]). The Henyey–Greenstein functions Eq. (20), have $N = \infty$, $\omega_n = (2n + 1)g^n$.

Table V shows such a comparison for conservative scattering. The phase function in each column is defined by $\omega_n = (2n + 1)(\frac{1}{2})^n$ for $n \leqslant N$ and 0 for $n > N$. The columns $N = 0$ and $N = \infty$ were copied from Tables III and IV. As Chandrasekhar has shown, the azimuth-independent terms in conservative scattering for $N = 1$ are identical to those for $N = 0$. The numbers for $N = 2$ have been computed by interpolation to $\omega_2 = 1.25$ in the tables of Horak and Janousek

TABLE V

RESULTS FOR TRUNCATED PHASE FUNCTIONS COMPARED WITH THOSE FOR THE FULL HENYEY–GREENSTEIN PHASE FUNCTION. THIS EXAMPLE REFERS TO CONSERVATIVE SCATTERING, $g = 0.50$

Function	$N = 0$ (isotropic)	$N = 1$ (linearly anisotropic)	$N = 2$	$N = \infty$ (full phase function)
$K(0)$	.4330	.4330	.3959	.3951
$K(0.5)$	.8716	.8716	.8642	.8691
$K(1)$	1.2591	1.2591	1.2742	1.2653
KN	.8660	.8660	.8579	.8599
$R(1, 0)$	.7270	.7270	.5182	.5774
$R(1, 0.5)$	.9755	.9755	.9118	.9581
$R(1, 1)$	1.0569	1.0569	1.1824	1.0950

[20]. The values of the escape function for $N = 2$ approximate those for the untruncated phase function within a fraction of one per cent. The differences in the reflection function seems more erratic, probably because the radiation that has been scattered only once or twice contributes a larger fraction of the total result. The maximum difference produced by truncating the phase function from $N = \infty$ to $N = 2$ in this example is 8 per cent.

A further comparison was made with a solution made by Weinman [21] for the radiative transfer of red light through a cloud of waterdrops with a prescribed size distribution. Weinman represents the conservative phase function by a gaussian diffraction peak plus 11 Legendre functions ($N = 10$) and solves the transfer problem by an approximate method. His curve representing the transmitted intensity for a total layer thickness 20 is quite smooth and agrees within 2 per cent with the curve computed by Twomey et al. [22] for the same situation. Assuming that $b = 20$ is thick enough for the asymptotic theory to hold we have renormalized this radiation to flux 1 and obtain the numbers

$\mu =$	1	0.9	0.7	0.5	0.3	0.1	0
$K(\mu) =$	1.35	1.21	1.02	0.86	0.67	0.47	0.37

with the average value [NK] = 0.84. These numbers agree indeed within a few per cent with those given in Table III, except for an overshoot of some 6 per cent at $\mu = 1$ and 10 per cent at $\mu = 0$. It is impossible to tell whether these differences must be attributed to reading numbers from the graph, or to the approximate nature of Weinman's computation, or to an actual difference between the functions $K(\mu)$ belonging to the underlying phase functions. But the suggestion is strong that the latter difference is very small indeed and would often vanish within the practical accuracy attained in astrophysical measurements.

A comparison of the albedos given by Weinman is equally gratifying. The flux integral of Eqs. (3b) and (4b) is

$$\mathrm{UT} = 1 - \mathrm{UR} = \frac{4K(\mu_0)}{3(1-g)(b+2q_0)} \tag{26}$$

We can take account of the diffraction peak by inserting $g = 0.734$ and $b = 0.570\tau^*$, where τ^* is the total depth in Weinman's calculation. Assuming q_0 known we obtain from Weinman's values of UR for $\tau^* = 8$ and 20 values of $K(\frac{1}{2})$ which differ by 0.3 per cent and 0.0 per cent, respectively, from the value in Table III.

Similar checks on even more asymmetric phase functions did not come out quite so well. The curves of Figure 2a from Irvine [11] are based on a phase function resembling that of a cumulus cloud at 0.7μ. Here $g = 0.8175$. Integrating these curves I found values of $1 - \mathrm{UR}$ which fall 5 to 10 per cent below the values predicted by Eq. (26) with $K(1)$ interpolated from Table III. The Monte-Carlo computations of Plass and Kattawar [23] show a difference in the same direction of 12 per cent and (in one case) 27 per cent. It is quite possible that $K(\mu)$ is somewhat different in these examples. Also, the reduced thickness $b(1-g)$, which is $\leqslant 4$ in the last example, may not be big enough yet to give a good approach to the asymptotic theory.

The final conclusion is that, in spite of the smooth character of the phase functions underlying Tables II to IV we can often use the results from these tables as a very good approximation to the radiative transfer in actual clouds. Practical formulae based on this conclusion and describing the reflection and transmission by very thick layers with nearly conservative scattering are presented elsewhere [24].

Acknowledgment

The computations for this paper were carried out at the N.A.S.A. Institute for Space Studies at New York by Dr. K. Grossman.

References

1. S. Chandrasekhar, "Radiative Transfer." Oxford Univ. Press, Oxford (1950).
2. V. V. Sobolev, "A Treatise on Radiative Transfer." Van Nostrand, Princeton, New Jersey (1963).
3. K. M. Case and P. F. Zweifel, "Linear Transport Theory." Addison-Wesley Publ. Cy, Reading (1968).
4. T. A. Germogenova, *J. Computational Math.* **1**, 1001 (1961).
5. H. C. van de Hulst, *Bull. Astr. Inst. Neth.* **20** (1968), in press.
6. J. L. Carlstedt and T. W. Mullikin, *Astrophys. J. Suppl.* **12**, 449 (1966).
7. N. J. McCormick, "Developments in Transport Theory," (E. Inönü and P. Zweifel, ed.), 323. Academic Press, New York (1967).
8. R. Aronson, *Nucl. Sci. Engineering* **27**, 271 (1967).
9. H. C. van de Hulst and W. M. Irvine, Mémoires Soc. R. Sc. Liège, Ve Série, Tome **7**, 78 (1963).
10. S. Twomey, H. Jakobowitz, and H. B. Howell, *J. Atmospheric Sci.* **23**, 289 (1966).
11. W. M. Irvine, *Astrophys. J.* **152**, 823 (1968).
12. H. C. van de Hulst and K. Grossman, "The Atmospheres of Venus and Mars," (Brandt and McElroy, ed.) (1968), in press.
13. G. H. Peebles and M. S. Plesset, *Phys. Rev.* **81**, 430 (1951).
14. D. W. N. Stibbs and R. E. Weir, *Mon. Notices Roy. Astr. Soc.* **119**, 512 (1959).
15. H. C. van de Hulst and F. G. Terhoeve, *Bull. Astr. Inst. Neth.* **18**, 377 (1966).
16. M. A. Heaslet and R. F. Warming, *Astrophysics Space Sci.* **1**, 460 (1968).
17. W. M. Irvine, *Astrophys. J.* **142**, 1563 (1965).
18. F. R. Vanmassenhove and C. C. Grosjean, "Electromagnetic Scattering," (Powell and Stein, ed.), 721. Gordon and Breach, New York (1967).
19. H. G. Horak and S. Chandrasekhar, *Astrophys. J.* **134**, 45 (1961).
20. H. G. Horak and A. L. Janousek, *Astrophys. J. Suppl.* **11**, 277 (1965).
21. J. A. Weinman, Icarus **9**, 67 (1968).
22. S. Twomey, H. Jakobowitz, and H. B. Howell, *J. Atmosheric Sci.* **24**, 70 (1967).
23. G. N. Plass and G. W. Kattawar, *Applied Optics* **7**, 361 (1968).
24. R. E. Danielson, D. R. Moore, and H. C. van de Hulst (1968). In preparation.